SEVENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Industries, and the Michigan Actuarial Society

NOVEMBER 14, 1963

PART I

INSTRUCTIONS
(To be read aloud to class by Supervisor or Proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer card.

2. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics. However, essentially all of the problems require a significant amount of computation. You are advised to do whatever figuring is necessary on scratch paper before selecting your answer.

3. Your score on the test will be the number right minus one-fifth the number wrong or left unanswered. If you cannot determine the right answer but are able to eliminate some of the alternatives as impossible, you are advised to guess an answer from the remaining alternatives.

4. The average score will probably be close to 10, with only about 4% of all participants scoring above 18 or 20. Thus you should not be discouraged if you find many problems too difficult. To improve your score, be careful to complete all problems which you can do successfully before spending time on more involved problems.

5. Your card will be graded by machine. Please read and follow carefully the instructions printed on the card. Do not make calculations on the answer card.

6. DO NOT PRINT YOUR NAME IN THE SPACE PROVIDED unless your name is incorrectly spelled at the top of your answer card or your name is not already on the card.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of this competition. If you have questions concerning the instructions, ask them now.

PART I

1. In order for $6\frac{1}{2} - x$ to be less than $2\frac{1}{3} + 1\frac{5}{6}$, x must be

   (A) less than $\frac{28}{12}$

   (B) greater than $\frac{7}{3}$

   (C) less than $-\frac{14}{6}$

   (D) greater than $-\frac{14}{6}$

   (E) none of the above

2. The expression $-2[3-(2-a)]$ $-5(1-a)$ is equal to

   (A) $-13-4a$ (B) $5+3a$

   (C) $-7+3a$ (D) $-13-6a$

   (E) $-15+3a$

3. One root of the equation $x^2 - 2x - 8 = 0$ is

   (A) the reciprocal of the other
   (B) the square of the other
   (C) twice the other
   (D) two more than the other
   (E) the negative of the other

4. Simplify: $(x^2 - \frac{1}{x^2})(1 + \frac{1}{x^2})^{-1}$

   (A) $x^4 - 1$

   (B) $(x - 1)(x + 1)$

   (C) $x^2 - \frac{1}{x^2} + 1 - \frac{1}{x^4}$

   (D) $1 - x^2$

   (E) none of the above

5. If the numerator of a fraction equivalent to $\frac{\sqrt{6}}{5 - \sqrt{6}}$ is 6, then its denominator is

   (A) -9 (B) 9

   (C) $6 - 5\sqrt{6}$ (D) $5\sqrt{6} - 6$

   (E) -1

6. Mr. Powell has \$40,000 invested at 8 per cent per year. He plans to invest more money at 6 per cent per year. He wants the combined annual income from both investments to be \$5,000. In order for this to be the case, his total investment T is such that

   (A) $T < \$40,000$

   (B) $\$40,000 \leq T < \$60,000$

   (C) $\$60,000 \leq T < \$80,000$

   (D) $\$80,000 \leq T$

   (E) none of the above

7. The sum of two numbers is 10. The sum of their reciprocals is 5. Their product is

   (A) $\frac{1}{2}$ (B) 2

   (C) 5 (D) 50

   (E) none of the above

8. A circle is inscribed in a triangle having sides 6, 8, and 10. The area of the circle is

   (A) 4π (B) $\pi/4$

   (C) $9\pi/2$ (D) 16π

   (E) none of the above

9. The sum of two irrational numbers is

(A) never an irrational number
(B) never a rational number
(C) never less than both of the numbers
(D) never an integer
(E) none of the above

10. The sum of 5 terms in an arithmetic progression is -5 and the 6th term is -13. If d is the common difference, $d^2 + \frac{1}{d}$ is

(A) $\frac{26}{3}$ (B) $\frac{63}{4}$
(C) $\frac{335}{28}$ (D) 0
(E) none of the above

11. If no two sides of a triangle have the same length, then the triangle is called a scalene triangle. The number of scalene triangles with all sides of integral length and perimeter less than 15 is

(A) 5 (B) 6
(C) 7 (D) 8
(E) 41

12. The length of the sides of a triangle are a + b, a - b, and $2\sqrt{ab}$. The triangle is

(A) acute
(B) obtuse
(C) right
(D) equilateral
(E) none of the above

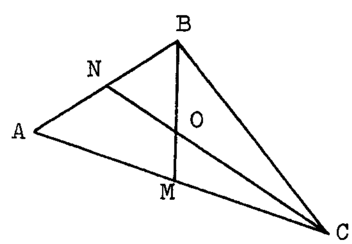
13. The intersection of a tetrahedron and a plane can

(A) never be a point
(B) never be a line segment
(C) never be a triangle
(D) never be a quadrilateral
(E) none of the above

14. The number of feet of wire needed to reach from the top of the pole to a stake on the ground 20 feet from the foot of the pole, if the wire forms an angle of 35° with the ground, is

(A) 20 cos 35°
(B) 20 sin 35°
(C) 20 sec 35°
(D) 20 csc 35°
(E) 20 tan 35°

15.



AM = MC, AN = NB. If ON = OB = 2, then OM + OC =

(A) 3 (B) 4
(C) 5 (D) 6
(E) none of the above

16. For two positive numbers, a and b, a is less than b and a is 12.5 per cent of the difference between a and b. What per cent is a of a + b?

(A) 8 (B) 10 (C) 12 (D) 14

(E) none of the above

17. A particle moves along the parabola $y = x^2 + 2x - 3$ from a point with y - coordinate zero to another point with y - coordinate zero. The horizontal distance moved is

(A) 2 (B) 3 (C) 4 (D) 5

(E) none of the above

18.
$$\begin{vmatrix} x & 1 & 2 \\ 0 & x & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

The roots of this equation add up to

(A) 0 (B) 1

(C) 4 (D) $2 + \sqrt{3}$

(E) none of the above

19. The distance from the point (a - b, a + b) to the point (a, b) is equal to the distance from the origin to the point

(A) (a - b, a)

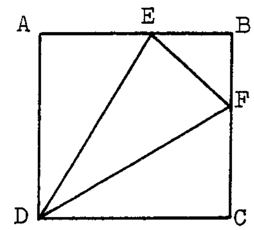
(B) (a - b, b)

(C) (a, a + b)

(D) (b, a + b)

(E) (-a, -b)

20.



ABCD is a square of side 1. DE and DF trisect $\angle ADC$. The area of triangle DEF is

(A) $\frac{\sqrt{3}}{3}$ (B) $\frac{2}{3}$

(C) $\frac{1}{2}$ (D) $\frac{1}{3}$

(E) none of the above

21. A function f is defined by $f(x) = a10^{bx}$ for all real numbers x. a and b are positive and $f(b) = 10^9 a$. Then

(A) b = 27

(B) b = 9

(C) b = 2 log 3

(D) b = log 3

(E) b = 3

22. Where $i = \sqrt{-1}$, which of the following does not represent a real number?

(A) $i + \frac{1}{i} + \frac{1}{i^2}$

(B) $i + i^2 + i^3$

(C) $(1 + i)^4$

(D) $i + i^3 + i^5$

(E) all of the above are real numbers

23. How many positive integers less than 1000 are not divisible by either 2, 3, or 5?

(A) 235 (B) 266

(C) 299 (D) 177

(E) none of the above

24. Two roots of

$2x^3 - 3x^2 + ax + b = 0$

are $3i$ and $-3i$. Find the value of b.

(A) 18 (B) 27/2

(C) -27 (D) 9

(E) none of the above

25. Point C is on segment AB with AB = 3CB. Circles are described on AC and CB as diameters and a common tangent meets the line containing AB at D. Then BD has length equal to

(A) the radius of the small circle

(B) the radius of the large circle

(C) the diameter of the large circle

(D) the sum of the radii of the two circles

(E) none of the above

26. A regular octagon is formed by cutting right triangles from the corners of a square of side 2. The area of the octagon is

(A) $2\sqrt{2}$ (B) $4\sqrt{2} - 2$

(C) $4\sqrt{2}$ (D) $8(\sqrt{2} - 1)$

(E) none of the above

27. Two candles of the same height are each consumed in three hours. If the first is lighted at seven o'clock and the second at eight o'clock, when is the second four times as tall as the first?

(A) 9:00 (B) 9:20

(C) 9:30 (D) 9:40

(E) none of the above

28. An equation whose roots are twice the roots of

$x^2 + bx + c = 0$ is

(A) $2x^2 + 2bx + 2c = 0$

(B) $x^2 + 2bx + 2c = 0$

(C) $x^2 + 2bx + 4c = 0$

(D) $x^2 + 4bx + 4c = 0$

(E) $x^2 + 4bx + 8c = 0$

29. A right circular cylinder and a sphere have the same radius. The total surface area of the cylinder will equal the surface area of the sphere in case

(A) its height equals its radius

(B) its height equals twice its radius

(C) its height equals one-half its radius

(D) its height equals four times its radius

(E) none of the above

30. Four points, A, B, C, and D, in space are not coplanar and are equidistant from one another. The line joining the point midway between A and B and the point midway between C and D is

(A) parallel to AC

(B) perpendicular to AC

(C) parallel to AB

(D) perpendicular to AB

(E) none of the above

31. Given that $0 < \log b - \log a < 1$, you can conclude that

(A) $1 < b - a < 10$

(B) $1 < b < 10$

(C) $1 < a < 10$

(D) $b < 10a$

(E) $a < b < 1$

32. A radioactive material decays at a rate such that at the end of any minute there is .75 as much as there was at the beginning of the minute. If we start with g grams, the number of minutes required for 90 per cent of this material to decay is

(A) $\frac{1}{\log 4 - \log 3}$

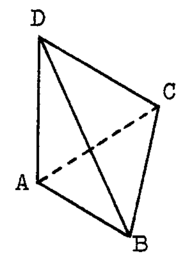
(B) $\frac{1}{\log 3 - \log 4}$

(C) $\log 3 - \log 4$

(D) $\log 4 - \log 3$

(E) none of the above

33.



ABCD is a tetrahedron.

$\angle CAD = \angle BAD = 90^\circ$.

$AB = BC = CA = \sqrt{3}$. $AD = 1$.

Then the dihedral angle between faces ABC and BDC is

(A) 30°

(B) 45°

(C) Arc sin $\frac{2}{3}$

(D) Arc tan $\frac{2}{3}$

(E) none of the above

34. If A runs x times as fast as B, in a 100-yard dash A will beat B by

(A) 100 x yards

(B) $\frac{100(x-1)}{x}$ yards

(C) $100(1 - x)$ yards

(D) $\frac{100}{x}$ yards

(E) none of the above

35. An equation of the line perpendicular to $3x - 2y - 7 = 0$ and passing through the point $(1, -1)$ is

(A) $2x + 3y + 1 = 0$
(B) $3x - 2y - 5 = 0$
(C) $2x + 3y + 5 = 0$
(D) $3y - 2x - 5 = 0$
(E) $3x + 2y - 1 = 0$

36. A club with 15 members forms x committees such that (1) each member is on exactly two committees and (2) each pair of committees has exactly one member in common. Then x =

(A) 4 (B) 5 (C) 6 (D) 8

(E) none of the above

37. $\sin [\text{Arc sin } \frac{1}{2} + \text{Arc cos } \frac{2}{3}]$ is equal to

(A) $\frac{2 + \sqrt{15}}{6}$

(B) $\frac{3 + 2\sqrt{5}}{6}$

(C) $\frac{9\sqrt{3} + 8\sqrt{5}}{36}$

(D) $\frac{\sqrt{5} + 2\sqrt{3}}{6}$

(E) none of the above

38. If $a - \frac{1}{a} > 0$ and $(a - \frac{1}{a})^2 = 3$, then $a^3 - \frac{1}{a^3} =$

(A) 0 (B) $\frac{26}{9}\sqrt{3}$

(C) $3\sqrt{3}$ (D) $6\sqrt{3}$

(E) $3 + \sqrt{3}$

39. If $1 + \frac{2 + \frac{6}{x}}{x + 3}$ does not differ from $\frac{1}{x}$ by more than $\frac{1}{10}$, then

(A) x does not differ from -1 by more than $\frac{1}{10}$

(B) x does not differ from -1 by more than $\frac{1}{11}$

(C) x does not differ from $\frac{-100}{99}$ by more than $\frac{10}{99}$

(D) x does not differ from $\frac{-99}{100}$ by more than $\frac{10}{99}$

(E) x does not differ from -1 by more than $\frac{10}{99}$

40. For a function f we know that $f(ax) = a f(x)$ for all real numbers a and x. From this it follows that

(A) $f(x) = \frac{f(a)}{a} x$, whenever $a \neq 0$

(B) $f(x) = x$

(C) $f(x) = \frac{f(a)}{a} x^2$, whenever $a \neq 0$

(D) $f(x) = ax$

(E) $\frac{f(x)}{a} = \frac{x}{f(a)}$, whenever $a \neq 0$ and $f(a) \neq 0$

The following Michigan companies have made contributions to the scholarship fund for this year's competition:

Aeroquip Corporation, Jackson
Calumet and Hecla, Incorporated, Allen Park
Clark Equipment Company, Battle Creek
Lear Siegler, Incorporated, Grand Rapids
Misco Precision Casting Company, Whitehall
Whitman and Barnes, Plymouth

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.