TENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America,
Michigan Colleges and Universities, Professional Organizations, and Industries.

PART I

OCTOBER 25, 1966

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out your ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right. You are advised to guess an answer in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible.

4. The average participant will have less than ten correct answers. To improve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of this competition. If you have questions concerning the instructions, ask them now.

1. The product of the roots of the equation $3x^2 + 9x = 2$ is equal to:

(A) 6.
(B) -2.
(C) $-\frac{2}{3}$.
(D) -3.
(E) None of these.

2. The negation of the statement, "All triangles are isosceles.", is:

(A) No triangles are isosceles.
(B) All triangles are scalene.
(C) Some triangles are scalene.
(D) No triangles are scalene.
(E) None of these.

3. If $x^{13} + x + 2$ is divided by $x + 1$, the remainder is:

(A) 0.
(B) 1.
(C) -1.
(D) 2.
(E) -2.

4. The graph of $y = \log x^2$, $x>0$,

(A) intersects all lines perpendicular to the x-axis.
(B) intersects all lines perpendicular to the y-axis.
(C) intersects neither co-ordinate axis.
(D) intersects all circles whose centers are at the origin.
(E) None of these.

5. The numeral in the units place when 7^{23} is expressed in base 12 notation is:

(A) 7.
(B) 3.
(C) 1.
(D) 11.
(E) None of these.

6. The expression $\frac{2^{n+3} - 2(2^n)}{2(2^{n+2})}$ is equal to:

(A) $\frac{3}{4}$.
(B) $2^{n+1} - \frac{1}{4}$.
(C) $\frac{3}{8}$
(D) -2^{n+1}.
(E) $1-2^n$.

7. The points in the xy-plane whose coordinates satisfy both of the conditions $x-y = 1$ and $x^2 + y^2 \leq 16$ constitute a set which consists of:

(A) only two points.

(B) an arc of a circle.

(C) a straight line segment.

(D) a single point.

(E) None of these.

8. The base of a given triangle is twice as long as the side of a given square, while the figures have equal areas. Then the ratio of the altitude of the triangle to the side of the square is:

(A) $\frac{1}{4}$.

(B) $\frac{1}{2}$.

(C) 1.

(D) 2.

(E) 4.

9. The expression $(x + y)^{-1}(x^{-1} + y^{-1})$ can be rewritten as:

(A) $x^{-2} + 2x^{-1}y^{-1} + y^{-2}$.

(B) $x^{-1}y^{-1}$.

(C) $x^{-2} + 2^{-1}x^{-1}y^{-1} + y^{-2}$.

(D) $x^{-2} + y^{-2}$.

(E) $\frac{1}{x^{-1}y^{-1}}$.

10. The coordinates of points P,Q, and R are (-2,-3), (4,1), and (0,k), respectively. The value of k that minimizes PR+RQ is:

(A) $\frac{5}{3}$.

(B) $-\frac{5}{3}$.

(C) $-\frac{5}{2}$.

(D) $\frac{5}{2}$

(E) None of these.

11. The sum of two irrational numbers is:

(A) never an irrational number.

(B) never a rational number.

(C) never less than both of the numbers.

(D) never an integer.

(E) None of these.

12. The arithmetic mean of a set P of ten numbers is p, and the arithmetic mean of a set Q of twenty numbers is q. Then the arithmetic mean of the set of all thirty numbers is:

(A) $\frac{(p + q)}{2}$.

(B) $\frac{(2p + q)}{3}$.

(C) $\frac{(p + 2q)}{3}$.

(D) $\frac{(p + q)}{3}$.

(E) None of these.

13. If the triangle $\triangle PQR$ is inscribed in a semicircle having $\overline{PQ}$ as diameter, then PR+QR must be:

(A) equal to PQ.

(B) equal to $PQ\cdot\sqrt{2}$.

(C) greater than or equal to $PQ\cdot\sqrt{2}$.

(D) less than or equal to $PQ\cdot\sqrt{2}$.

(E) None of these.

14. If the determinants $\begin{vmatrix} x & 1 & 3 \\ 2 & 1 & 3 \\ 2 & 3 & 2 \end{vmatrix}$ and $\begin{vmatrix} 2 & 1 & 3 \\ x & 1 & 3 \\ 0 & 3 & 2 \end{vmatrix}$ are equal then x equals:

(A) $-\frac{1}{2}$.

(B) -2 .

(C) $\frac{1}{2}$.

(D) 2.

(E) None of these.

15. A trial consists of drawing a ball from a bag of balls and then discarding the ball. A trial is a success if a red ball is drawn. If the bag initially contains exactly five red and ten white balls, then the probability of at least one success in the first two trials is:

(A) $\frac{3}{7}$

(B) $\frac{4}{7}$

(C) $\frac{1}{2}$

(D) $\frac{5}{9}$

(E) None of these.

16. The integral roots of the set of equations, $\{x^z = y^{2z}$, $3(3^z) = 3^x$, $x-y + z = 9\}$ are:

(A) x = 4, y = -3, z = 2.

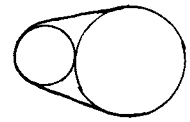
(B) x = 2, y = -3, z = 4.

(C) x = 3, y = -2, z = 4.

(D) x = 4, y = -2, z = 3.

(E) None of these.

17. Two steel poles having diameters of 12 and 36 inches respectively are placed side-by-side as in the cross-sectional figure and bound together with wire. The length of the shortest piece of wire that will go around the resulting bundle is:



(A) $24\sqrt{3} + 32\pi$.

(B) $24\sqrt{3} + 14\pi$.

(C) $24 + 28\pi$.

(D) $24\sqrt{3} + 28\pi$.

(E) $24\pi + 48$.

18. A value of $\arccos(\sin \frac{\pi}{3})$ is:

(A) $\frac{\pi}{3}$

(B) $\frac{-\pi}{3}$

(C) $\frac{\pi}{6}$

(D) $\frac{-\pi}{6}$

(E) None of these.

19. Three circles, two with the same radius R and one with radius r, are drawn on the same side of a straight line in such a way that each circle is tangent to the line and both other circles. The relationship between R and r is:

(A) $R = 4r$.

(B) $R = 5r$.

(C) $R = 6r$.

(D) $R = 3r$.

(E) None of these.

20. Each face of a regular tetrahedron is colored either red, green, or blue. How many distinct colorings of the tetrahedron are possible, independent of position?

(A) 3^4.

(B) 12.

(C) 4^3.

(D) 18.

(E) None of these.

21. The radius of a cylindrical can is 8 inches and its height is 2 inches. The number x of inches that can be added either to the radius or to the height so that the resulting positive increase in volume is the same in both cases, is:

(A) 1.

(B) 2.

(C) $\frac{5}{2}$.

(D) 3.

(E) None of these.

22. If $f(x) = \frac{1}{1 - x}$, $x \neq 0$ and $x \neq 1$, then $f(f(f(x))) =$

(A) x.

(B) $\frac{1}{x}$

(C) $1 - x$.

(D) $f(x)$.

(E) None of these.

23. If the three points of contact of a circle inscribed in a triangle are joined, then the resulting triangle always has:

(A) angles equal in measure.

(B) one obtuse angle and two unequal acute angles.

(C) one obtuse angle and two equal acute angles.

(D) three acute angles.

(E) unequal angles.

24. For what real x is $\frac{|x - 1|}{|x + 3|} < 1$?

(A) $-1 < x < 1$.

(B) All real $x \neq -3$.

(C) $-3 < x < 1$.

(D) All $x < -1$, $x \neq -3$.

(E) All $x > -1$.

25. If the point P lies in the plane outside of an equilateral triangle ΔRST of side s, and if P is joined to each point on ΔRST by a straight line segment, then the locus of the midpoints of these line segments is:

(A) a line segment parallel to a side of ΔRST.

(B) a circle.

(C) an equilateral triangle of side $\frac{s}{3}$.

(D) an equilateral triangle of side $\frac{s}{2}$.

(E) None of these.

26. Let P be an interior point of the regular tetrahedron T. Then the sum s of the distances from P to the vertices of T is never less than d, the sum of the distances of P from the faces of T, times:

(A) 3.

(B) $\sqrt{8}$.

(C) 4.

(D) 1.

(E) None of these.

27. The number of solutions of the equation $8\sin^4 x + 8\cos^4 x = 5$ in the interval $0 < x < 2\pi$ is:

(A) 2.

(B) 4.

(C) 6.

(D) 8.

(E) None of these.

28. N is an integer such that N divided by 10 leaves the remainder 9, N divided by 9 leaves the remainder 8, N divided by 8 leaves the remainder 7, etc. In other words, N divided by n leaves the remainder $n - 1$ for every integer n satisfying $1 < n \leq 10$. Which of the following conclusions about $N + 1$ is false?

(A) $N + 1$ is divisible by 8,9, and 10.

(B) $N + 1$ is divisible by 12,20, and 28.

(C) $N + 1$ is divisible by 6,15, and 21.

(D) $N + 1$ is divisible by 4,12, and 20.

(E) $N + 1$ is divisible by 9,18, and 27.

29. Four spheres of the same radius R have the property that each is tangent to the other three spheres. Then the minimal radius possible for a fifth sphere which is tangent to each of the given four is:

(A) $R\left(2\left(\frac{2}{3}\right)^{\frac{3}{2}} - 1\right)$.

(B) $\frac{R}{\sqrt{3}}$.

(C) $R\left(\sqrt{\frac{3}{2}} - 1\right)$.

(D) $\frac{R}{4}$.

(E) None of these.

30. The condition $\frac{x + 3}{x + 1} \geq x$ is equivalent to the condition:

(A) $x \leq -\sqrt{3}$.

(B) $-\sqrt{3} \leq x \leq \sqrt{3}$.

(C) $x \geq \sqrt{3}$.

(D) $-1 < x \leq \sqrt{3}$.

(E) None of these.

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this year's competition.

Aeroquip Foundation, Jackson
Electro-Voice, Buchanan
Long Manufacturing, Detroit
Packaging Corporation, Filer City
Misco Precision Casting Company, Muskegon

The names of other companies, contributing to the scholarship fund during the next few months, will be reported in later announcements.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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