TWENTY-FIFTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

PART 1

October 14, 1981

INSTRUCTIONS
(to be read aloud to the students by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to insure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in ovals completely.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer. Usually a score of about 20 will allow you to become a finalist and write Part II of the competition.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "(e) none of these." If you believe none of the first four alternatives to be correct, mark E, in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request any one to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You may now open the test booklet and begin.

25th ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

1. The graphs of $y = |x|$ and $y = \sin x$ intersect in

   (a) no points (b) 1 point (c) 2 points (d) 3 points

   (e) more than 3 points

2. If N is an integer and 13N is a multiple of 5 then N

   (a) is a multiple of 65 (b) is odd (c) is even (d) is a multiple of 5

   (e) is none of the above

3. For $x \neq 0$ and $x \neq -1$, the expression $1 + \dfrac{1}{1 + \frac{1}{x}}$ is identical to

   (a) $1 - \dfrac{x}{x+1}$ (b) $2 + \dfrac{1}{x+1}$ (c) $\dfrac{2x+1}{x+1}$ (d) $\dfrac{x+2}{x+1}$ (e) none of these

4. If $f(2) = 3$ and $g(3) = 2$ then $f(g(3))$ is

   (a) 6 (b) 2 (c) 3 (d) 18 (e) none of these

5. The solution of $\dfrac{x-1}{x+2} > 1$ is given by

   (a) $x > -2$ (b) $x < -2$ (c) $-2 < x < 1$ (d) the empty set

   (e) none of these

6. If one of the dimensions (base or height) of a triangle is shortened by a certain percentage and the other dimension is lengthened by the same percentage, what can you say about the area?

   (a) It always gets larger. (b) It always gets smaller.

   (c) It gets larger if and only if the height is increased.

   (d) It gets larger if and only if the base is increased.

   (e) The area remains the same.

7. The sum of all the roots of the polynomial $ax^4 + 3x^3 + 4x^2 - 2x + b$

(a) depends on a and b (b) depends on b but not a (c) is 3 (d) is 0

(e) is $-3/a$

8. If two socks are drawn at random from a drawer containing 6 grey socks and 8 blue socks, then the probability of getting a matched pair is

(a) 12/49 (b) 86/91 (c) 24/49 (d) 28/91 (e) 43/91

9. $\left(\sqrt{4 + \sqrt{4 + \sqrt{4}}}\right)^4$ equals

(a) 100 (b) 192 (c) $22 + 8\sqrt{6}$ (d) $14 + 4\sqrt{6}$ (e) 52

10. If $\sin x = \cos y = 1/3$ and x, y are acute then $\sin (x + y)$ is

(a) $2\sqrt{8}/9$ (b) 0 (c) 1 (d) 2 (e) 7/9

11. If $\sin\left(\frac{2\pi}{9}\right) \cos\left(\frac{2\pi}{9}\right) = \frac{1}{2} \cos\left(\frac{3\pi}{2} + x\right)$ then x, in radians, could equal

(a) $\pi/18$ (b) $5\pi/18$ (c) $4\pi/9$ (d) $2\pi/9$ (e) 0

12. In a class of 60 junior and senior students, 21 are girls and 34 are juniors and 20 are senior boys. How many juniors are girls?

(a) 15 (b) 19 (c) 6 (d) 20 (e) 8

13. If $n = 7$ then $\log\left(\frac{n}{n+1}\right) + \log\left(\frac{n+1}{n+2}\right) + \ldots + \log\left(\frac{2n-1}{2n}\right)$ is identical to

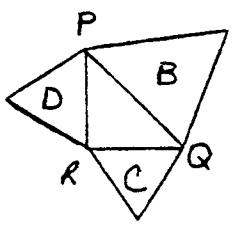
(a) $-\log (1/2)$ (b) $-\log 49$ (c) $-\log 7$ (d) $-\log 2$

(e) none of these

14. In the diagram, PQ is the hypotenuse of isosceles right triangle PQR and B, C and D are the areas of equilateral triangles with one side of each common with a distinct side of PQR. Then it follows that, as areas,

(a) $B = D + C$ (b) $B = \sqrt{2}(D + C)$ (c) $B = (D + C)/\sqrt{2}$

(d) $B = \sqrt{2}\, C$ (e) none of these



15. The equation $\frac{1}{x-3} + \frac{1}{x-7} = \frac{3x-17}{(x-3)(x-7)}$ has the following number of solutions

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

16. Which of the following is a fourth power of an integer ?

(a) 156,250,004 (b) 155,226,682 (c) 158,728,728 (d) 154,160,510

(e) 157,351,936

17. All solutions $x = c$ of $2x^{-2} + 3x^{-3} = 4x^{-4}$ satisfy

(a) $0 < c < 4$ (b) $-1 < c < 3$ (c) $-2 < c < 2$ (d) $-3 < c < 1$

(e) none of these is true

18. An isosceles right triangle has the same area as an equilateral triangle with side length one. The length of the hypotenuse of the right triangle is:

(a) $\sqrt{3}/2$ (b) $\sqrt{3}$ (c) $\sqrt[4]{3}$ (d) $\sqrt[4]{3/2}$ (e) none of these

19. A radiator contains 20 liters of a mixture which is 40% antifreeze. You want to drain some off and add an equal volume of pure antifreeze to bring the percentage up to 60%. How many liters should you drain off?

(a) 4 (b) 16/3 (c) 20/3 (d) 8 (e) 12

20. The point which divides the segment from (-4,-3) to (5,3) in the ratio of 5 to 3 has coordinates

(a) (13/8,6/8) (b) (7/5,3/5) (c) (-5/8,-6/8) (d) (-2/5,-3/5)

(e) none of these

21. Given two fixed points A and B in a plane, for a third point P in the plane to be placed so that angle APB is acute, it is necessary and sufficient that:

(a) P lies on the perpendicular bisector of AB.

(b) P lies outside of a certain band bounded by lines parallel to AB.

(c) P lies inside the circle having line segment AB as a diameter.

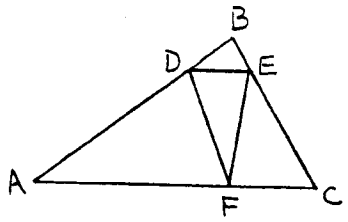
(d) P lies outside of a square having two sides parallel to AB and A and B are the midpoints of the other two sides.

(e) none of these

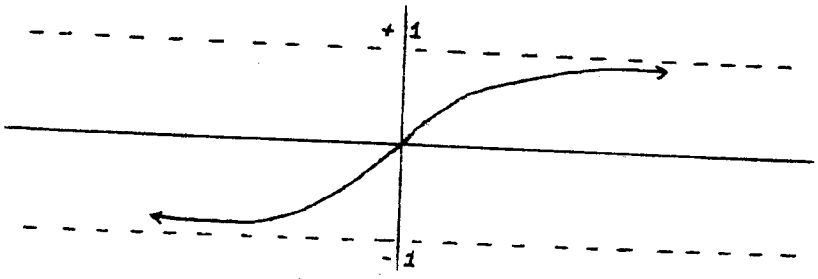
22. In the triangle pictured, the ratios AD:DB and CE:EB are both 4:1. F is an arbitrary point on AC. Then the ratio of the area of triangle DEF to the area of triangle ABC is:

(a) 1:4 (b) 4:25 (c) 1:5 (d) 3:16

(e) not enough information about F is given to solve this.



23. The graph pictured could be the graph of the function given by



(a) $f(x) = \arctan(x)$ (b) $f(x) = \frac{x}{x+1}$ (c) $f(x) = \frac{x}{|x|+1}$

(d) $f(x) = \frac{x}{x^2+1}$ (e) $f(x) = \sqrt[3]{x}$

24. A geometric progression with positive terms has its second term equal to 5 and its eighth term equal to 17. Then the first term and the common ratio are, respectively

(a) $5(5/17)^{\frac{1}{6}}$ and $(17/5)^{\frac{1}{6}}$ (b) $5(5/17)^{\frac{1}{3}}$ and $(17/5)^{\frac{1}{6}}$ (c) 3 and 2

(d) $5(5/15)^{\frac{1}{7}}$ and $(17/5)^{\frac{1}{7}}$ (e) none of these

25. The inequality $|x^2 - 5| \geq 2$ is satisfied if and only if

(a) $x \geq 7$ or $-3 \leq x \leq 3$ (b) $x \geq \sqrt{7}$ or $x \leq -\sqrt{7}$ or $-\sqrt{3} \leq x \leq \sqrt{3}$

(c) $x \geq \sqrt{7}$ or $0 \leq x \leq \sqrt{3}$ (d) $x \geq \sqrt{7}$ (e) $x \leq -\sqrt{7}$ or $x \geq \sqrt{7}$

26. Two of the roots of $x^4 + ax^3 + bx^2 + cx + d$ are $3-2i$ and $5+i$ and all of a, b, c and d are real numbers. Then the coefficient d is

(a) 17-7i (b) 16 (c) 17+7i (d) 8 (e) 338

27. Given digits r, s, and t with $9 \geq r > s > t \geq 1$, which of the products of two integers is greatest. Note: rs = 10r + s, etc.

(a) $t \cdot rs$ (b) $s \cdot rt$ (c) $t \cdot sr$ (d) $r \cdot st$ (e) $r \cdot ts$

28. Given the $\log_{10} 2 = a$ and $\log_{10} 3 = b$, then $\log_{10} 15$ is

(a) $b(a+b)$ (b) $1 + b - a$ (c) $2b + 3a$ (d) $3(a+b)$

(e) cannot be determined with the given information

29. If $3^x = (27)^{y-5}$ and $2^y = 8^{2x-4}$ then $y - x$ equals

(a) 3 (b) 9 (c) 5 (d) 2 (e) 4

30. The area of a circle is 36 square feet; an equilaterial triangle is inscribed in it. What is the area of the triangle (in square feet)?

(a) $\frac{27\sqrt{3}}{\pi}$ (b) $\frac{18\sqrt{3}}{\pi}$ (c) $\frac{9\sqrt{6}}{\pi}$ (d) $\frac{9\sqrt{3}}{\pi}$ (e) $\frac{9\sqrt{6}}{2\pi}$

31. $\sqrt{6 + 3\sqrt{3}} - \sqrt{6 - 3\sqrt{3}}$ equals

(a) 6 (b) 9 (c) $\sqrt{63}$ (d) 3 (e) $\sqrt{6}$

32. The longer leg of a right triangle is one foot less than twice the shorter leg. The hypotenuse of the triangle is 17 feet long. The area of the triangle (in square feet) is:

(a) 45 (b) $17\sqrt{2}$ (c) 120 (d) 90 (e) 60

33. The units digit in $(8193)^{237}$ is

(a) 9 (b) 3 (c) 7 (d) 5 (e) 1

34. The sum of all the numerical coefficients in the expansion of $(4x - 3y)^{15}$ is

(a) (-1) (b) 7^{15} (c) $4^{15} - 3^{15}$ (d) 1 (e) $4^{15} + 3^{15}$

35. Quadratic polynomial p(x) has remainder 1 on division by x - 1 or x - 2. The product of the two roots is 1. Then the sum of the two roots is

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

36. A two digit number is divided by the sum of its digits; then the quotient is 2 and the remainder is 2. If the number is multiplied by the sum of its digits, the product is 112. Then the product of its digits is

(a) 6 (b) 5 (c) 4 (d) 3 (e) 12

37. Joe, alone can paint a garage in 24 hours and his brother Bobby, alone, can paint the same garage in 12 hours. The two boys and their father, together, can paint the garage in 4 hours. How long would it take the father alone to paint the garage (in hours)?

(a) 4 (b) 3 (c) 8 (d) 2/3 (e) 3/2

38. If the recurring decimal .630630630... is written as the fraction p/q in its simplest form then q-p is

(a) 936 (b) 36 (c) 4 (d) 41 (e) 94

39. If $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{14}{k} = \binom{14}{k-2}$ then $\binom{k}{6}$ is

(a) 91 (b) 13 (c) 28 (d) not defined (e) 56

40. Suppose that the following three statements are assumed to be true:

(i) All guys who wear green hats are good guys.

(ii) Some guys don't wear hats.

(iii) All guys are good or bad but not both.

Which of the following statements logically follows from these?

(a) All good guys who wear hats wear green hats.

(b) All guys who wear nongreen hats are bad.

(c) Some good guys wear no hats.

(d) All bad guys who wear hats wear nongreen hats.

(e) Some guys who wear nongreen hats are bad.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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