TWENTY-SEVENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

PART I

October 19, 1983

INSTRUCTIONS

(to be read aloud to the students by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to insure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer. Usually a score of about 20 will allow you to become a finalist and write Part II of the competition.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "(E) none of these". If you believe none of the first four alternatives to be correct, mark E, in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request any one to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You may now open the test booklet and begin.

27TH ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

1. If $x = \sqrt{\frac{1}{9} + \frac{1}{4}}$, the closest approximation to x among the following is:

   (A) 0.36 (B) $\frac{5}{6}$ (C) 0.6 (D) $\frac{1}{3}$ (E) $\frac{\pi}{10}$

2. Let f be the function defined by $f(x) = 5 - x + 2x^2$. If $f(m) = 1$ then:

   (A) $m > 5$ (B) $m < -5$ (C) m is rational (D) m is irrational

   (E) none of these

3. If $x^{17} + 1$ is divided by $x + 1$, then the remainder is:

   (A) 1 (B) -1 (C) 0 (D) $2x$ (E) none of these

4. If the number x satisfies $8 + x < 4 - 3x < 5x - 4$, then:

   (A) $x < -1$ (B) $1 < x$ (C) $-1 < x < 1$ (D) $-2 < x < 2$

   (E) there is no such number

5. The equation $x + \sqrt{x - 2} = 4$ has

   (A) 2 real roots (B) 1 real and 1 imaginary root (C) no roots

   (D) 1 real root (E) 2 imaginary roots

6. A certain laboratory animal weighed 150 grams at age 5 days and 380 grams at age 30 days. Assuming a constant daily rate of weight gain during this period, what was its weight to the nearest tenth of a gram at age 18 days?

   (A) 228.0 (B) 269.6 (C) 274.0 (D) 315.6 (E) 347.6

7. y is proportional to x^2, and y is 8 when x is 8. When x is 6, y is:

   (A) 2 (B) $2\frac{1}{2}$ (C) 3 (D) 6 (E) not listed

8. Two cubic dice are thrown. The probability of getting 5 as the sum of the spots on the two top faces is:

(A) $\frac{1}{9}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{4}$ (E) not listed.

9. A parallelogram has all its vertices on a circle. Which of the following is true?

(A) It has perpendicular diagonals. (B) It is a rhombus.

(C) It has equal diagonals. (D) It is a square.

(E) There is no such parallelogram.

10. If $32^{(3x-1)} = 4^{(x+1)}$ then x is:

(A) $\frac{7}{13}$ (B) 1 (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\log_4 32$

11. If $\log_{10} 5 = A$ and $\log_{10} 8 = B$ then:

(A) $A = 1 - \frac{B}{3}$ (B) $B = 1 - \frac{A}{2}$ (C) $A > B$ (D) $A = 3B - 1$ (E) $B - A = \log_{10} 3$

12. In the (simplified) binomial expansion of $(x + x^{-2})^{22}$ the number of terms in which x appears with a negative exponent is:

(A) 5 (B) 15 (C) 11 (D) 14 (E) not listed.

13. In the diagram $\angle\alpha = 27^\circ$ and $\angle\beta = 37^\circ$. Then the measure of angle γ is:

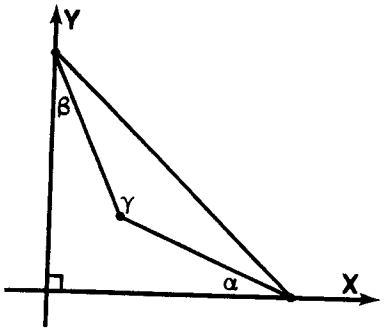
(A) 116°

(B) 154°

(C) 104°

(D) 114°

(E) none of these.



14. The number -1 has three cube roots. The square of one of them is:

(A) $1 + i$ (B) $\sqrt{3} + 2i$ (C) $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ (D) $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$ (E) none of these.

15. A set has five elements. The total number of its subsets is:

(A) 5 (B) 10 (C) 25 (D) 31 (E) not listed.

16. The set of points in the coordinate plane such that $x^2 - y^2 \geq 0$ and $|x| \leq 1$:

(A) is the null set.

(B) consists of 2 parallel line segments.

(C) consists of 2 circular sectors.

(D) covers a region of area 2.

(E) is unbounded.

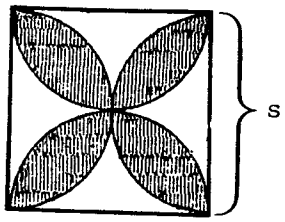
17. If $\frac{1}{x} - \frac{1}{y} = 1$ and $2x + y = 7xy$, then $x + y$ is:

(A) $\frac{5}{6}$ (B) 1 (C) $\frac{6}{5}$ (D) $\frac{2}{3}$ (E) not listed.

18. The four-leafed figure shown is obtained by drawing semicircles on each side of a square with side length s. The area (shaded region) of this figure is:

(A) $\frac{\pi}{4}s^2$ (B) $\frac{4 - \pi}{2}s^2$ (C) $\frac{1}{3}s^2$

(D) $\frac{1}{2}s^2$ (E) $(\frac{\pi}{2} - 1)s^2$



19. If $u_n = \frac{1}{2 + u_{n-1}}$ and $u_o = 0$ then u_5 is:

(A) 0.4 (B) 0.5 (C) $\frac{12}{29}$ (D) $\frac{11}{30}$ (E) not listed.

20. If the points A, B, C in the coordinate plane have coordinates (0, 0), (1, 2), (3, 1) respectively then the area of ΔABC is:

(A) 2 (B) $\frac{5}{2}$ (C) $\sqrt{15}$ (D) $\frac{7}{2}$ (E) not listed.

21. In a certain town 60% of the families own a dog and 40% own a cat. Which of the following statements must be false?

(A) All families own a cat or a dog.
(B) All cat owners also own a dog.
(C) More than half the dog owners also own a cat.
(D) All dog owners also own a cat.
(E) If the number of dog owners increased by 10% some families would have to own a dog and a cat.

22. A triangle has side lengths 3, 4, and 5. The radius of its inscribed circle is:

(A) 2 (B) $\frac{\sqrt{2}}{2}$ (C) 1 (D) $\frac{\sqrt{15}}{4}$ (E) not listed.

23. The radian measure, to the nearest hundredth, of a 75° angle is:

(A) 1.31 (B) 1.42 (C) 0.75 (D) 0.83 (E) 1.57

24. The graph of $x^2 + y^2 - 2x - 4y + 5 = 0$ is

(A) a circle with center at (2, 1).
(B) a single point.
(C) a circle of radius $\sqrt{5}$
(D) the null set.
(E) not listed.

25. If $\begin{vmatrix} x & 1 & 2 \\ 4 & 2 & 4 \\ a & 3 & 1 \end{vmatrix} = 0$ then x is:

(A) 2 (B) 4 (C) a (D) a + 4 (E) not listed.

26. Consider the following six real functions of x :

(1) x (2) $\sqrt{2}\,x^2 + \pi$ (3) $\frac{1}{2}$

(4) $x + x^{-2}$ (5) 2^x (6) $x^{\frac{1}{2}}$

Among these, the polynomials are those numbered:

(A) 1, 2, 3 (B) 1, 2, 4 (C) 2, 3, 4, 6 (D) 1 (E) not listed.

27. The repeating decimal 0.545454... is equal to a fraction in lowest terms whose numerator is:

(A) 23 (B) 18 (C) 33 (D) 54 (E) not listed.

28. A set of points in the plane is *convex* if the line segment joining any two of its points is also in the set. Which of the following sets in the coordinate plane is <u>not</u> convex?

(A) $\{(x, y) \mid x^2 + y^2 < 1\}$ (B) $\{(x, y) \mid x^2 + y^2 = 0\}$

(C) $\{(x, y) \mid x = y\}$ (D) $\{(x, y) \mid xy > 0\}$

(E) $\{(x, y) \mid y \geq x^2\}$

29. When the number given as 2222 in base 10 is converted to base 5, its leftmost and rightmost digits are, respectively:

(A) 4 and 2 (B) 1 and 1 (C) 3 and 2 (D) 4 and 4 (E) 2 and 2

30. If a is a negative number then the number of positive roots of the equation $x^5 - x + a = 0$ is:

(A) 0 or 2 depending on a (B) 2 (C) 1

(D) 1 or 3 depending on a (E) 3

31. Two nonzero numbers are such that three times their sum is two more than their product, and their difference is one less than their average. One of the numbers is:

(A) 2/3 (B) 3 (C) 17/3 (D) 10 (E) 47/3

32. Two pumps are available to empty an oil storage tank. Used separately the first could empty it in 6 hours and the second could empty it in 8 hours. If the pumps are used together the time it takes in hours, to the nearest tenth, is:

(A) 3.2 (B) 3.4 (C) 4.0 (D) 5.8 (E) 7.0

33. A committee is to be formed containing 3 men and 4 women. If the available candidates consist of 12 men and 16 women then the number of possible different committees is:

(A) 400,400 (B) 410,310 (C) 375,000 (D) $\frac{28!}{7!}$ (E) not listed

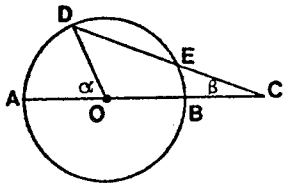
34. If we define a glub to be a parallelogram with equal perpendicular diagonals which share a common midpoint then:

(A) the definition is inconsistent. (B) some but not all glubs are squares.

(C) the definition is redundant. (D) the class of glubs is empty.

(E) any rhombus is a glub.

35. In the figure, the circle with center O has radius 1 and $\overline{EC} = 1$. Then:



(A) $\alpha = 2\beta$

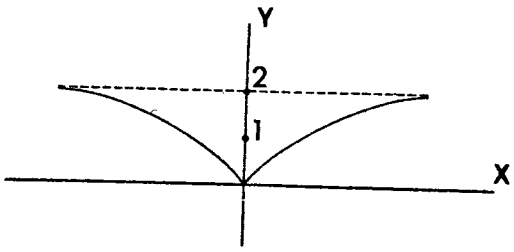
(B) $\alpha = 1$ radian

(C) $\alpha = 3\beta$

(D) $\sin \alpha = 2 \sin \beta$

(E) $2\beta < \alpha < 3\beta$

36. Which of the following functions could have the graph pictured?



(A) $y = \dfrac{2x^2}{|x| + 1}$

(B) $y = \dfrac{|x|}{|x| + 1}$

(C) $y = \dfrac{2x^2}{|x + 1|}$

(D) $y = \left|\dfrac{2x}{x + 1}\right|$

(E) not listed.

37. You are asked to construct triangle ABC satisfying: $\overline{BC} = 3.5$, $\overline{AC} = 7.2$ and $\angle A = 31^\circ$. Which of the following is true?

(A) $\angle B$ is less than $\angle A$.

(B) There is no such triangle.

(C) Side AB is the longest.

(D) Two non-congruent triangles satisfy the requirements.

(E) It is a right triangle.

38. If $\tan 2x = \sin x$ and $0 < x < \dfrac{\pi}{2}$ then:

(A) $x = \dfrac{\pi}{4}$

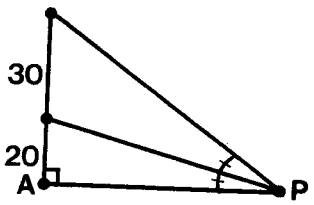
(B) $\dfrac{\pi}{4} < x < \dfrac{\pi}{3}$

(C) there is no such x

(D) $\tan 2x = \dfrac{3}{4}$

(E) $\cos^2 x = \sin x$

39. A pole 50 feet tall has a white band 20 feet above the ground. A point P on the ground from which the parts above and below the band subtend equal angles satisfies:



(A) $\overline{AP} = 50$

(B) $\overline{AP} = 40$

(C) $\overline{AP} = 10\sqrt{10}$

(D) $\overline{AP} = 20\sqrt{5}$

(E) There is no such point.

40. If $\dfrac{2x + 1}{(x - 2)(x - 1)^2} = \dfrac{Ax + B}{x - 2} + \dfrac{Cx + D}{(x - 1)^2}$ where A, B, C and D are constants, then $A + B + C + D$ is:

(A) 2 (B) 3 (C) -1 (D) 1 (E) There are no such numbers.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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