

**THIRTY-NINTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION**

Sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 11, 1995

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.
3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine an answer.
5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of the others". If you believe none of the first four alternatives to be correct, mark e) in such cases.
6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes or any other aid is prohibited. If you have questions concerning the instructions, ask them now.
7. You may now open the test booklet and begin.

1. The expression

$$\frac{1 - \frac{x-2}{x+2}}{\frac{x+2}{x-2} - 1}$$

simplifies to

- a) 0 b) 1 c) $\frac{x+2}{x-2}$ d) $\frac{x-2}{x+2}$ e) none of these
2. Find the value of x in radians to the nearest 0.001, if it is known that $\sin x = 0.345$ and $\frac{\pi}{2} < x < \pi$.
- a) 2.789 b) 20.182 c) 1.923 d) -0.352 e) 0.352
3. The difference in the areas of two concentric circles is equal to five times the area of the smaller circle. The ratio of their radii is
- a) $\sqrt{6}$ b) $\sqrt{5}$ c) 6 d) 5 e) π
4. In the expression $\frac{\pi}{2} + \frac{2}{x} + \frac{\pi}{2} + \frac{2}{x} + \frac{\pi}{2} + \frac{2}{x} + \dots$ the terms are alternately $\frac{\pi}{2}$ and $\frac{2}{x}$. What is the sum of the first 72 terms of the expression?
- a) $\frac{18x^2+36}{x}$ b) $\frac{36x^2+72}{x}$ c) $\frac{9x^2+18}{x}$ d) $\frac{9x^2+36}{x}$ e) $\frac{18x^2+72}{x}$
5. Integer a is less than five times integer b , but seven times integer a is greater than seven times integer b . One-fifth of integer a is greater than integer c . Which one of the following can we conclude?
- a) $c < a < b$ b) $a > c > b$ c) $\frac{a}{5} < c < \frac{b}{7}$ d) $c < b < a$ e) none of these
6. The liquid in a full conic tank with radius 10 feet and height 50 feet fits perfectly into ten identical spherical containers. What is the radius of each spherical container?
- a) $\frac{5}{2}\sqrt[3]{12}$ ft. b) $\frac{5}{3}\sqrt[3]{36}$ ft. c) 5 ft. d) $5\sqrt[3]{2}$ ft. e) $5\sqrt[3]{4}$ ft.
7. The size of a TV screen is described by the length of the diagonal. The ratio of height to width is the same for all televisions. How much larger is the area of a 30" screen than the area of a 20" screen?
- a) the areas are the same b) the ratio of the areas is 3:2
c) the ratio of the areas is 2:1 d) the ratio of the areas is 9:4
e) the ratio of the areas is 4:1

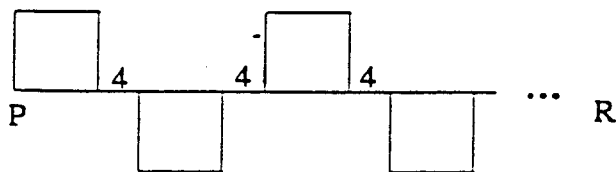
8. Positive integers $x, a, b,$ and c satisfy the equation

$$1 - \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}} = \frac{1}{x + \frac{1}{a + \frac{1}{b + \frac{1}{c}}}}$$

Find x .

- a) $x = 1$ b) $x = 2$ c) $x = 3$ d) $x = 4$
e) two or more values of x satisfy the equation
9. $ABCDEF$ is a hexagon with the property that each side, AB, BC, CD, DE, EF, FA has length 10, angles FAB and CDE are right angles and the other four angles, ABC, BCD, DEF, EFA are equal in measure. The area of the hexagon is
- a) less than 100 b) between 100 and 200 c) 200
d) between 200 and 250 e) more than 250
10. If $16 \sin x = 32 \cos 2x - 33$, then $\sin x =$
- a) $1/8$ b) $-1/8$ c) $1/16$ d) $-1/16$ e) $31/16$
11. A square is inscribed in a circle of area 72π . Find the length of a side of the square.
- a) 6 b) 12 c) $6\sqrt{2}$ d) $6\sqrt{2}$ e) not enough information is given
12. Nine pebbles are placed in a three by three grid. How many ways can three pebbles be picked so that no two of them are in the same row or column?
- a) 1 b) 2 c) 3 d) 4 e) 6
13. A Nautical mile is defined to be the distance subtended by one minute of arc on the equator. The distance around the equator is roughly 25000 miles. Roughly how long is a Nautical mile?
- a) 0.86 mi b) 1 mi c) 1.16 mi d) 1.35 mi e) 2.52 mi
14. The diagonal of a rectangle is 10. The area of the rectangle
- a) must be less than 40 b) must be between 40 and 50 c) must be 50
d) cannot be over 50
e) can be determined precisely from the above information

15. The figure below shows a pattern of squares, each of area 64, separated by segments of length 4 along line segment PR . If the pattern continues to the right, at which of the following distances from P will the total areas of the squares above PR be equal to the total area of the squares below PR ?



- a) 56 b) 60 c) 70 d) 80 e) 84
16. A collection of 10 numbers ranges from a low of 5 to a high of 17 and has an average of 9. If the high and the low are discarded, the remaining collection of 8 numbers has an average of
- a) 7 b) between 7 and 8 c) 8 d) between 8 and 9 e) more than 9
17. The angles of a triangle are in arithmetic progression. If the longest side of the triangle has length 5 and the shortest side has length 1, the area of the triangle is
- a) 3 b) $5\sqrt{3}/4$ c) $5\sqrt{2}/4$ d) 4 e) $5\sqrt{3}/3$
18. A marksman fired at a target which had five areas worth scores of 16, 17, 23, 24, and 39 points respectively. His total was exactly 100. The number of times that the area worth 16 points was hit is:
- a) 0 b) 1 c) 2 d) 3 e) 4
19. When the integers from 1 to 1995 are added, a certain sum S is obtained. For what integer N is it true that the sum of the integers from 1996 to N is closest to S ?
- a) 2495 b) 2661 c) 2742 d) 2822 e) 2994
20. Given $\triangle ABC$ with $BC = 1$, $AC = 2$, and $\angle C = 30^\circ$. Which of the following is *not* true?
- a) $\angle B = 2\angle A$ b) $AB = \sqrt{5 - 2\sqrt{3}}$ c) $\sin B = 2 \sin A$ d) $\angle A$ is less than 30°
 e) $\angle B$ is obtuse

21. Sixty items are in a box. Some of them are silly, some are crazy, and some are nutty. All the nutty items are also crazy; no nutty items are silly; nine of the items are both silly and crazy; fifteen items are crazy but not nutty or silly; a total of twenty-one items are silly; and seventeen items are neither silly nor crazy. How many items are nutty?
- a) 7 b) 9 c) 12 d) 13 e) 16
22. A quadrilateral plot of land $ABCD$ is almost rectangular. Surveyors measure the length of the sides AB and AD and the length of the diagonal BD . The diagonal is used to check how rectangular the plot is. The number
- $$d = \frac{BD^2 - AB^2 - AD^2}{2AB \cdot AD}$$
- is used to check how square the angle A is. The plot is considered rectangular if $|d| < 0.05$. What does it tell you about the angle A in degrees?
- a) $|A - 90| < 3$ b) $|A - 45| < 0.05$ c) $|A - 90| < 0.05$ d) $|A - 90| > 5$
e) none of these
23. Those points in the first quadrant of the xy -plane which have both coordinates perfect squares of positive integers are called "illuminated". How many illuminated points are inside the circle of radius 27 centered at the origin?
- a) 20 b) 22 c) 24 d) 26 e) 28
24. If $f(0) = 1$ and $f(1) = 5$, and for positive integers $n \geq 2$, $f(n) = 2f(n - 1) - f(n - 2) + 2$, then $f(10) =$
- a) 109 b) 129 c) 131 d) 155 e) none of these
25. Given a cube of side s , let d represents the length of a diagonal of one of its faces and D represents the length of a main diagonal of the cube, i.e. a diagonal connecting two opposite vertices of the cube. Which of the following are true:
I. $3s < D$ II. $s + d < D$ III. $2D^2 = 3d^2$
- a) I only b) II only c) III only d) I and II only e) II and III only
26. The vertices of triangle ABC have coordinates $A(-1, 2)$, $B(2, -2)$, and $C(-1, -6)$. Triangle ABC is
- a) equilateral b) isosceles with base 5 c) isosceles with base 8
d) right scalene e) acute scalene

27. If $\log_{10} 7 = \ln x$, then the value of x to the nearest .001 is

- a) .401 b) 0.700 c) 1.903 d) 2.328 e) 5.178

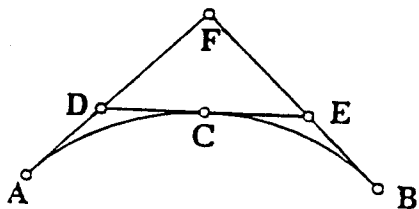
28. What are the possible number of points of intersection of a circle and a square?

- a) 1, 2, 3, 4 b) 0, 1, 2, 3, 4, 5, 6, 7, 8 c) 1, 2, 3, 4, 8 d) 0, 1, 2, 4, 6, 8
e) 1, 2, 3, 4, 5, 6, 7, 8

29. $\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots \sqrt{1}}}}$ (1000 of the $\sqrt{\quad}$ symbols) is closest to

- a) 1.4 b) 1.5 c) 1.6 d) 1.7 e) 1.8

30. In the diagram, arc ACB is a quarter-circular arc of radius 1 with $AC = CB$. Line segments DE , FA , and FB are tangent to the arc at points C , A , and B respectively. Find the area of triangle DEF .



- a) $\frac{4-\pi}{4}$ b) $3 - \sqrt{8}$ c) $\frac{1}{8}$ d) $\frac{\pi-\sqrt{2}}{8}$ e) $\frac{\pi-2}{8}$

31. Suppose that N is the least positive integer such that $44N$ is a perfect cube. Then:

- a) $N \leq 100$ b) $100 < N \leq 300$ c) $300 < N \leq 500$ d) $500 < N \leq 1000$
e) $N > 1000$

32. In triangle ACD , point B is located on the side AC so that $AB = BD$ and $BC = CD$. If angle ACD is 80° , what is the measure, in degrees, of angle CAD ?

- a) 15 b) 20 c) 25 d) 30 e) 35

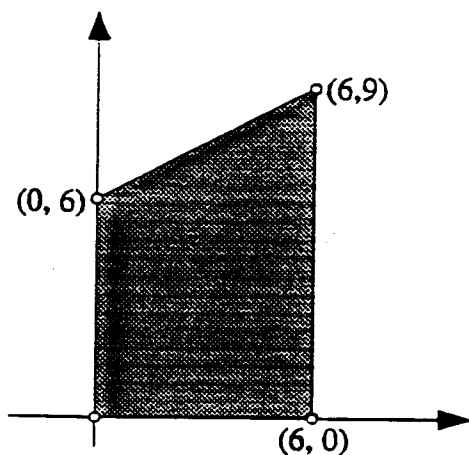
33. The three stooges, Larry, Moe, and Curly, make fruity purchases. Larry buys one banana, two oranges, and two apples for \$ 2.00. Moe buys three bananas, one orange, and one apple for \$ 1.50. Which of the following purchases by Curly would enable you to compute the cost of an apple?
- a) one banana for \$ 0.20 b) one orange and one apple for \$ 0.90
c) four bananas, three oranges, and three apples for \$ 3.50
d) one banana, one orange, and two apples for \$ 1.60
e) two bananas, three oranges, and three apples for \$ 3.10
34. How long must a driver take to drive the last 50 miles of a 120 mile trip if he wants to average 50 mph for the entire trip and it took 90 minutes to drive the first 70 miles?
- a) 54 min. b) 1 hr. c) 66 min d) 70 min. e) 75 min.
35. On June 1, Jill earns a certain amount of money. Each following day of June, July, and August she earns one dollar more than she did on the previous day. By August 31, she has been paid a total of \$ 6670. How much did she earn on the Fourth of July?
- a) \$ 58 b) \$ 59 c) \$ 60 d) \$ 61 e) \$ 72
36. 1995^{1995} is a huge number. How many digits does it have?
- a) 3990 b) 3991 c) 6583 d) 6584 e) more than 10000
37. The roots of the equation $x^3 + 6ax^2 + ax + 30$ are all integers. Which of the following is a possibility for the roots of the equation?
- a) 3, 1, -10 b) 3, -2, 5 c) 3, 2, -5 d) -3, 2, -5 e) 3, -3, 6
38. The formula for monthly payment P on a loan is

$$P = \frac{Ai(1 + \frac{i}{12})^n}{12[(1 + \frac{i}{12})^n - 1]}$$

where A is the amount borrowed, i is the annual percentage rate of interest on the loan, and n is the term (number of months to repay the loan). If \$ 44,000 is borrowed at A.P.R. (annual percentage rate) 9.25%, and the monthly payment is \$ 452.85, what was the term of the loan?

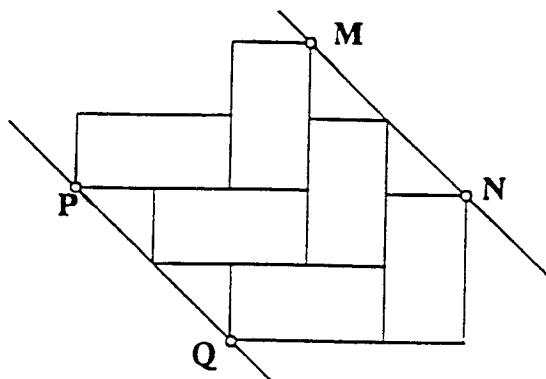
- a) 10 years b) 12 years c) 13 years d) 15 years e) 20 years

39. The center of gravity of the uniform metal plate pictured is



- a) (3.2, 2.8) b) (2.8, 3.8) c) (2.8, 3.2) d) (3.2, 6.1) e) (3.2, 3.8)

40. Rectangular bricks are laid in the manner shown below:



If each brick is 4 centimeters wide and 8 centimeters long, what is the perpendicular distance, in centimeters, between the lines MN and PQ ?

- a) 80 b) $10\sqrt{2}$ c) $4\sqrt{13}$ d) $8\sqrt{2}$ e) $6\sqrt{2}$

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Mathematical Association of America

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