Forty-Second Annual

# Michigan Mathematics Prize Competition

Sponsored by the
Michigan Section of the Mathematical Association of America

## Part II

December 9, 1998



1. An organization decides to raise funds by holding a \$60 a plate dinner. They get prices from two caterers. The first caterer charges \$50 a plate. The second caterer charges according to the following schedule: \$500 set-up fee plus \$40 a plate for up to and including 61 plates, and $\$2500 \log_{10}\left(\frac{p}{4}\right)$ for $p > 61$ plates.

   a) For what number of plates $N$ does it become at least as cheap to use the second caterer as the first?

   b) Let $N$ be the number you found in a). For what number of plates $X$ is the second caterer's price exactly double the price for $N$ plates?

   c) Let $X$ be the number you found in b). When $X$ people appear for the dinner, how much profit does the organization raise for itself by using the second caterer?

2. Let $N$ be a positive integer. Prove the following:

   a) If $N$ is divisible by 4, then $N$ can be expressed as the sum of two or more consecutive odd integers.

   b) If $N$ is a prime number, then $N$ cannot be expressed as the sum of two or more consecutive odd integers.

   c) If $N$ is twice some odd integer, then $N$ cannot be expressed as the sum of two or more consecutive odd integers.

3. Let $S = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$.

   a) Find, in terms of $S$, the value of $\dfrac{1}{2^2} + \dfrac{1}{4^2} + \dfrac{1}{6^2} + \dfrac{1}{8^2} + \cdots$.

   b) Find, in terms of $S$, the value of $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots$.

   c) Find, in terms of $S$, the value of $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots$.

4. Let $\{P_1, P_2, P_3, \ldots\}$ be an infinite set of points on the $x$-axis having positive integer coordinates, and let $Q$ be an arbitrary point in the plane not on the $x$-axis. Prove that infinitely many of the distances $|P_iQ|$ are not integers.

   a) Draw a relevant picture.

   b) Provide a proof.

5. Point $P$ is an arbitrary point inside triangle $ABC$. Points $X$, $Y$, and $Z$ are constructed to make segments $PX$, $PY$, and $PZ$ perpendicular to $AB$, $BC$, and $CA$, respectively. Let $x$, $y$, and $z$ denote the lengths of the segments $PX$, $PY$, and $PZ$, respectively.

   a) If triangle $ABC$ is an equilateral triangle, prove that $x + y + z$ does not change regardless of the location of $P$ inside triangle $ABC$.

   b) If triangle $ABC$ is an isosceles triangle with $|BC| = |CA|$, prove that $x + y + z$ does not change when $P$ moves along a line parallel to $AB$.

   c) Now suppose that triangle $ABC$ is scalene (i.e., $|AB|$, $|BC|$, and $|CA|$ are all different). Prove that there exists a line for which $x + y + z$ does not change when $P$ moves along this line.