FORTY-THIRD ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 13, 1999

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of these". If you believe none of the first four alternatives to be correct, mark e) in such cases.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include portable computers and pocket organizers. All problems will be solvable with no more technology than a scientific calculator. Possessing a more powerful calculator will not be a significant advantage.

7. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

1. The diameter of the star Castor is estimated to be 11 times the diameter of our own sun. What is the ratio of Castor's volume to the volume of our sun?

a) $(5.5)^2$ b) $(5.5)^3$ c) 11 d) 11^2 e) 11^3

2. Which is more likely with perfect six-sided dice:
(i) to roll a 1, then a 2, and then a 3 on consecutive rolls of a single die, or
(ii) to roll three identical dice and have the outcome be a 1, a 2, and a 3?

a) (i) is more likely
b) (ii) is more likely
c) (i) and (ii) are possible and equally likely
d) you can't tell from the information given
e) none of these

3. For which values of x is it true that $\dfrac{x-1}{x+1} > 2$?

a) $x < -3$ b) $-3 < x < -1$ c) $-1 < x < 3$ d) $-1 < x$ e) $-3 < x$

4. Suppose that $f(x)$ is defined and positive for all real numbers x.
If f increases for $x < 0$ and decreases for $x > 0$, and if $g(x) = \dfrac{1}{f\left(\frac{1}{x}\right)}$, then

a) g increases for all real x
b) g increases for $x < 0$ and decreases for $x > 0$
c) g decreases for all real x
d) g decreases for $x < 0$ and increases for $x > 0$
e) there is not enough information to determine where g increases or decreases

5. For each real number α, let $N(\alpha)$ denote the number of distinct real numbers x that satisfy the equation $x = \dfrac{\alpha}{1+x}$. How many different values does $N(\alpha)$ have?

a) 1 b) 2 c) 3 d) 4 e) more than 4

6. At summer camp, 10% of the participants failed the swimming test, while 15% failed the canoe-safety test. About 7% of the campers passed neither test. How many of those who passed swimming failed canoe-safety?

a) about 8% b) about 14% c) about 15% d) about 16.4% e) about 25%

7. Let $S = \{5, 10, 13, 15, 18, 23, 29, 30, 32, 38, 41\}$. Split S into two nonempty subsets A and B such that $S = A \cup B$, $A \cap B = \emptyset$, and $a - b$ is odd for every $a \in A$ and $b \in B$. How many different possibilities are there for the set A?

a) 11! b) 2^{11} c) 11 d) 2 e) none of these

8. How many solutions does the system of equations $x^2 + y^2 = 1$, $2x + 5y = 10$ have?

a) 0 b) 1 c) 2 d) 3 e) 4

9. Suppose $f(x) = x^2 - 2x$. Find $f(f(2i + 1))$.

a) -5 b) $5i + 5$ c) 15 d) 35 e) $15i - 10$

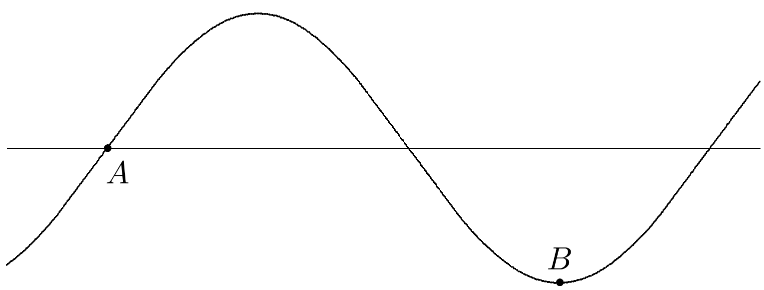
10. What is the equation of the straight line that passes through the point $(1, 2)$ and is parallel to the line $5x - 2y + 2 = 0$?

a) $5x - 2y - 1 = 0$ b) $2x - 5y + 8 = 0$ c) $5x + 2y - 9 = 0$
d) $5x - 2y + 1 = 0$ e) $2x + 5y - 12 = 0$

11. The Smiths are a family with two children. You learn that at least one of the two children is a boy (but you don't find out anything about which of the two is a boy). Based on this information, what is the probability that both of their children are boys?

a) 0 b) 1/4 c) 1/3 d) 1/2 e) 1

12.



The graph above is that of $y = 2\sin(cx - 1)$. Point A has coordinates $(1/c, 0)$. What is the x-coordinate of Point B?

a) $\frac{1}{c} + \frac{3\pi}{2}$ b) $\frac{1}{c} - \frac{\pi}{2c}$ c) $1 + \frac{3\pi}{2c}$ d) $\frac{1}{c} + \frac{3\pi}{2c}$ e) $1 + \frac{3\pi c}{2}$

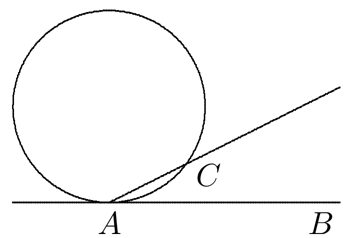
13. A balloon in the shape of a cube is being filled with gas. The balloon is made of a remarkable material that retains its cubical shape while being inflated. After 5 minutes, the volume is 1000 cubic feet. By how many square feet has the surface area increased when the volume of the balloon has grown from 1000 cubic feet to 2000 cubic feet?

a) $\sqrt{3} - 1$ b) $60(2^{3/2} - 1)$ c) $600(2^{2/3} - 1)$ d) $6 \cdot 300^{3/2} - 1$ e) none of these

14. At a gathering of n people ($n > 1$), each person shakes hands with some subset of the other people in attendance. Is it possible that no two people shake hands with the same number of other people?

a) No, that is not possible.
b) It is possible for all values of n as long as $n > 1$.
c) It is possible if n is sufficiently large.
d) It is possible if n is even, but impossible if n is odd.
e) It is possible if n is odd, but impossible of n is even.

15. In the figure to the right, $\angle BAC$ formed by the tangent line AB and chord AC has measure $20°$. If arc AC is 2 units long, which of the following is closest to the radius of the circle?



a) 2 b) 2.86 c) 3.14 d) 4.71 e) 5.73

16. For which values of x is it true that $|2x - 1| < 5$?

a) $x < 3$ b) $x < -2$ c) $-2 < x < 3$ d) $2 < x < 3$ e) $3 < x$

17. What rational number is represented by the repeating binary expansion $.\overline{1100}$ (where binary expansion $.x_1x_2x_3\cdots$ means $x_1 \cdot \frac{1}{2} + x_2 \cdot \frac{1}{4} + x_3 \cdot \frac{1}{8} + \cdots$)?

a) $\frac{110}{220}$ b) $\frac{3}{4}$ c) $\frac{4}{5}$ d) 1 e) none of these

18. A rectangular floor is tiled in a checkerboard fashion with square tiles of edge length 1. The dimensions of the floor are 321 units by 123 units. An insect walks along the diagonal of the rectangle. How many 4-corner intersection points does the insect cross on its walk between diagonally opposite corners?

a) 2 b) 40 c) 106 d) 122 e) 320

19. Let n denote a positive integer. Then $x + 1$ is a factor of $x^n - x - 2$ if

a) n is larger than 3 b) n is even c) n is a prime
d) n is any positive integer e) n is odd

20. If z is a complex number such that $\left(z + \frac{1}{z}\right)^2 = 3$, then $z^3 + \frac{1}{z^3}$ equals

a) $\frac{10\sqrt{3}}{3}$ b) $3\sqrt{3}$ c) 0 d) $7\sqrt{7}$ e) $6\sqrt{3}$

21. Four suits (diamonds, hearts, spades, clubs) of playing cards constitute a deck. Each suit has 13 cards. The cards are face down, and you pick up one card at a time. Your first four cards are diamonds. What is the probability that your fifth card will also be a diamond?

a) 1/4 b) 3/16 c) 1/52 d) 9/52 e) none of these

22. If you double the size of a sphere, which quantity increases by the larger (multiplicative) factor:

- the radius of the sphere or
- the ratio of the volume of the sphere divided by its surface area

a) the radius b) the ratio of the volume divided by the surface area
c) it depends on whether double means double the radius or double the volume
d) they both increase by exactly the same factor no matter which interpretation of double you use
e) none of these

23. The solution set of the equation $x^{10} = 10^x$ consists of

a) two negative numbers b) one negative and one positive number
c) one negative and two positive numbers
d) two positive numbers e) two negative and one positive number

24. Katie has taken three exams and received an average of 70% on them. Her scores on these exams will determine 60% of her course grade, while the score on the final exam will determine 40% of her course grade. What grade does she need to receive on her final exam in order to achieve an average of 80% in the course?

a) 75% b) 80% c) 85% d) 90% e) 95%

25. Suppose you randomly order the numbers $1, 2, \ldots, n$. What is the probability that 2 will come immediately after 1?

a) $\dfrac{1}{n-1}$ b) $\dfrac{1}{n}$ c) $\dfrac{1}{n(n-1)}$ d) $\dfrac{1}{n!}$ e) $\dfrac{1}{(n-1)!}$

26. A sequence $a_1, a_2, \ldots$ is defined by $a_1 = 3$ and $a_{n+1} = a_n^2 + a_n$. What is the next-to-last (second from the right) digit of the number a_{1999}?

a) 1 b) 3 c) 5 d) 7 e) 9

27. If $1 < x < 2$, then $y = \dfrac{x}{2} + \dfrac{1}{x}$ is fairly close to $\sqrt{2}$. For example, if $x = \frac{3}{2}$, then $y = \frac{17}{12} = 1.4166\cdots$. Which of the following statements best describes the values of y?

a) $\sqrt{2} \le y$ for all $x \in (1,2)$　　b) $\sqrt{2} \ge y$ for all $x \in (1,2)$
c) $\sqrt{2} < y$ for some $x \in (1,2)$ and $\sqrt{2} > y$ for some $x \in (1,2)$
d) $\sqrt{2} < y$ for all $x \in (1,2)$　　e) $\sqrt{2} > y$ for all $x \in (1,2)$

28. You take a hollow clear plastic cube and hold it so that one corner rests on a table with the opposite corner pointed directly toward the ceiling. You slowly fill the cube with water through a small opening near the top. As you do so, the surface of the water inside the cube forms geometric shapes. Which of the following describes the shapes you see?

a) at first a triangle, then a hexagon, then a triangle again
b) at first a triangle, then a square, then a rectangle
c) triangle from beginning to end
d) a rectangle from beginning to end
e) at first a rectangle, then a triangle, then a rectangle again

29. If x is a real number, then $(1 - |x|)(1 + x) > 0$ if and only if

a) $|x| < 1$　　b) $x < 1$　　c) $|x| > 1$　　d) $x < -1$　　e) $x < -1$ or $-1 < x < 1$

30. A standard deck of cards is shuffled and two cards are drawn. What is the probability that they are from the same suit?

a) 1/4　　b) 4/17　　c) 5/17　　d) 1/3　　e) none of these

31. What is the smallest number n so that no matter how you place n marks on a 3-by-3 tic-tac-toe board, you are assured of getting three in a row (either horizontally, vertically, or diagonally)?

a) $n = 3$　　b) $n = 4$　　c) $n = 5$　　d) $n = 6$　　e) $n = 7$

32. The measure of angle θ in degrees is $(50 - x)^\circ$. What is the measure of the complement of θ in radians?

a) $\dfrac{\pi}{180}(40 + x)$　　b) $\dfrac{\pi}{180}(40 - x)$　　c) $\dfrac{\pi}{360}(40 + x)$　　d) $\dfrac{\pi}{360}(40 - x)$　　e) none of these

33. The polynomial $p(x)$ has remainder 3 when divided by $x - 1$ and remainder 5 when divided by $x - 3$. What is its remainder when divided by $(x - 1)(x - 3)$?

a) $x - 2$　　b) $x + 2$　　c) 2　　d) 8　　e) 15

34. In base n, the number 25 is doubled when its digits are reversed. Which of the following sets contains the number n?

a) $\{5, 6, 7\}$ b) $\{8, 9, 10\}$ c) $\{11, 12, 13\}$ d) $\{14, 15, 16\}$ e) none of these

35. Let S be a square, let C be the circle inscribed in S, and let D be the circle circumscribed about S. Which statement most accurately describes the relationship between the areas of C and D?

a) $\dfrac{\text{area}(D)}{\text{area}(C)} = 2$ b) $\dfrac{\text{area}(D)}{\text{area}(C)} = 4$ c) $\dfrac{\text{area}(D)}{\text{area}(C)} = \dfrac{1}{2}$

d) $\dfrac{\text{area}(D)}{\text{area}(C)} = \text{area}(S)$ for every square S

e) $\dfrac{\text{area}(D)}{\text{area}(C)}$ cannot be determined from the information given.

36. What is the leftmost point on the curve $x = y^2 - y + 2$?

a) $\left(\frac{3}{4}, \frac{1}{2}\right)$ b) $\left(\frac{7}{4}, \frac{1}{4}\right)$ c) $\left(\frac{7}{4}, \frac{1}{2}\right)$ d) $\left(\frac{1}{2}, \frac{7}{4}\right)$ e) there is no such point

37. Suppose a, b, and c are real numbers. Which, if any, of the following statements about the equation $(x - a)(x - b) = c$ is true?

a) If $c > 0$, its roots are always real.
b) If $c > 0$, its roots are never real.
c) If $c < 0$, its roots are always real.
d) If $c < 0$, its roots are never real.
e) Statements a), b), c), d) above are false.

38. You have \$18.00 that you intend to spend eating dinner at a restaurant. Once you buy the food, you will have to pay 5% tax on the cost of the food, and you will leave a 15% tip on the cost of the food. Given that you cannot spend more than the \$18.00 that you have in your pocket, what is the most you could spend on food?

a) 15 dollars b) 18/(1.05)(1.15) dollars c) 12 dollars
d) 18/(2.20) dollars e) none of these

39. Write $(1 - i)^{10}$ as a complex number in the form $a + bi$.

a) $-32i$ b) $5 + \sqrt{7}\, i$ c) $5 - \sqrt{7}\, i$ d) 32 e) none of these

40. A 1-centimeter wide insulating ribbon is wrapped in spiral fashion on a very long cylindrical pipe so that the pipe is just covered with the ribbon (no overlap). The contractor calculates that 1500 cm of ribbon is used per meter of pipe. Which of the following is closest to the (outer) diameter of the pipe?

a) 2.39 cm b) 4.40 cm c) 4.77 cm d) 5.10 cm e) 5.45 cm

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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