MMPC STUDENT CODE

FORTY-EIGHTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 1, 2004

INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

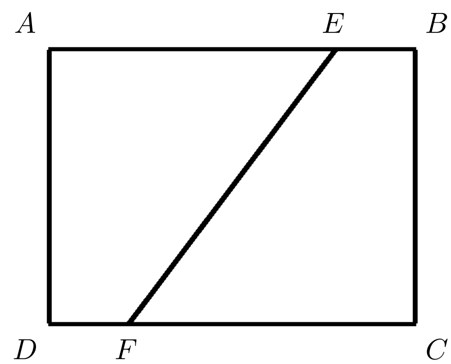
7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. The following figure represents a rectangular piece of paper $ABCD$ whose dimensions are 4 inches by 3 inches. When the paper is folded along the line segment $\overline{EF}$, the corners A and C coincide.

   (a) Find the length of segment $\overline{EF}$.

   (b) Extend $\overline{AD}$ and $\overline{EF}$ so they meet at G. Find the area of the triangle $\triangle AEG$.



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2. (a) Let p be a prime number. If a, b, c, and d are distinct integers such that the equation

$$(x-a)(x-b)(x-c)(x-d)-p^2=0$$

has an integer solution r, show that

$$(r-a)+(r-b)+(r-c)+(r-d)=0.$$

(b) Show that r must be a double root of the equation $(x-a)(x-b)(x-c)(x-d)-p^2=0$.

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3. If $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$, prove the following statements.

   (a) $\cos(x - y) = -\dfrac{1}{2}$

   (b) $\cos(\theta - x) + \cos(\theta - y) + \cos(\theta - z) = 0$, for any angle θ.

   (c) $\sin^2 x + \sin^2 y + \sin^2 z = \dfrac{3}{2}$

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4. Let $|A|$ denote the number of elements in the set A.

   (a) Construct an infinite collection $\{A_i\}$ of infinite subsets of the set of natural numbers such that $|A_i \cap A_j| = 0$ for $i \neq j$.

   (b) Construct an infinite collection $\{B_i\}$ of infinite subsets of the set of natural numbers such that $|B_i \cap B_j|$ gives a distinct integer for every pair of i and j, $i \neq j$.

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5. Consider the equation $x^4 + y^4 = z^5$.

    (a) Show that the equation has a solution where x, y, and z are positive integers.

    (b) Show that the equation has infinitely many solutions where x, y, and z are positive integers.

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(continued solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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